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Dynamical phase transition in the two-point functions of the autonomous one-dimensional single-species reaction-diffusion systems

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Abstract. The evolution of the two-point functions of autonomous one-dimensional single-species reaction-diffusion systems with nearest-neighbor interaction and translationally-invariant initial conditions is investigated. It is shown that the dynamical phase structure of such systems consists of five phases. As an example, a one-parameter family is introduced which can be in each of these phases.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.Ga Markov processes

1 Introduction

Reaction-diffusion systems, is a well-studied area. People have studied reaction-diffusion systems, using analytical techniques, approximation methods, and simulation. The approximation methods may be different in different dimensions, as for example the mean field techniques, working good for high dimensions, generally do not give correct results for low dimensional systems. A large fraction of analytical studies, belong to low-dimensional (specially one-dimensional) systems, as solving low-dimensional systems should in principle be easier [1–13].

The term exactly-solvable has been used with different meanings. In [14–16], integrability means that the N-particle conditional probabilities' S-matrix is factorized into a product of 2-particle S-matrices. In [17–26], solvability means closedness of the evolution equation of the empty intervals (or their generalization).

In [27], a ten-parameter family of reaction-diffusion processes was introduced for which the evolution equation of n-point functions contains only n- or less-point functions. We call such systems autonomous. The average particle-number in each site has been obtained exactly for these models. In [28,29], this has been generalized to multi-species systems and more-than-two-site interactions.

Among the important aspects of reaction-diffusion systems, is the phase structure of the system. The static phase structure concerns with the time-independent profiles of the system, while the dynamical phase structure concerns with the evolution of the system, specially its relaxation behavior. In [30–33], the phase structure of some classes of single- or multiple-species reaction-diffusion systems have been investigated. These investigations were bases on the one-point functions of the systems.

Here we want to study the two-point functions of autonomous single-species one-dimensional reaction-diffusion systems. Throughout this study, the initial condition of the system is taken to be translationally-invariant, so that it remains translational-invariant during the evolution. The two-point function for such systems is obtained, and it is shown that it exhibits a non-trivial dynamical phase structure. In Section 2, the evolution equation of the two-point function is obtained. In Section 3, this equation is solved and the corresponding energy-spectrum is obtained. In Section 4, the parameter space of the system is analyzed. In Section 5, the dynamical phase structure of the system (the different phase regions in the parameter space) is investigated. Finally, Section 6 is devoted to a one-parameter example family, which can be in all five phases.

2 Evolution equations of the one- and two-point functions

Consider a one-dimensional periodic lattice, every point of which is empty or contains one particle. Let the lattice have $L+1$ sites. The observables of such a system are the operators N_i^α, where i with $1 \leq i \leq L+1$ denotes the site number, and $\alpha = 0, 1$ denotes the hole or the particle: N_i^0 is the hole (vacancy) number operator at site i, and N_i^1 is

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the particle number operator at site i. One has obviously the constraint

$$s_\alpha N_i^\alpha = 1, \tag{1}$$

where $\mathbf{s}$ is a covector the components of which (s_α's) are all equal to one. The constraint (1), simply says that every site is either occupied by one particle or empty. A representation for these observables is

$$N_i^\alpha := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes N^\alpha \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L+1-i}, \tag{2}$$

where N^α is a diagonal 2×2 matrix the only nonzero element of which is the α'th diagonal element, and the operators 1 in the above expression are also 2×2 matrices. It is seen that the constraint (1) can be written as

$$\mathbf{s} \cdot \mathbf{N} = 1, \tag{3}$$

where $\mathbf{N}$ is a vector the components of which are N^α's. The state of the system is characterized by a vector

$$\mathbf{P} \in \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{L+1}, \tag{4}$$

where $\mathbb{V}$ is a 2-dimensional vector space. All the elements of the vector $\mathbf{P}$ are nonnegative, and

$$\mathbf{S} \cdot \mathbf{P} = 1. \tag{5}$$

Here $\mathbf{S}$ is the tensor-product of $L+1$ covectors $\mathbf{s}$.

As the values of the number operators N_i^α are zero or one (and hence N_i^α's are idempotent), the most general observable of such a system is the product of some of these number operators, or a sum of such terms. Moreover, the constraint (1) shows that the two components of $\mathbf{N}_i$ are not independent. so, one can express any function of $\mathbf{N}_i$ in terms of

$$n_i := \mathbf{a} \cdot \mathbf{N}_i, \tag{6}$$

where $\mathbf{a}$ is an arbitrary covector not parallel to $\mathbf{s}$. Our aim is to study the evolution of the two-point functions constructed by n_i's.

The evolution of the state of the system is given by

$$\dot{\mathbf{P}} = \mathcal{H}\, \mathbf{P}, \tag{7}$$

where the Hamiltonian $\mathcal{H}$ is stochastic, by which it is meant that its nondiagonal elements are nonnegative and

$$\mathbf{S}\, \mathcal{H} = 0. \tag{8}$$

The interaction is nearest-neighbor, if the Hamiltonian is of the form

$$\mathcal{H} = \sum_{i=1}^{L+1} H_{i,i+1}, \tag{9}$$

where

$$H_{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes H \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-i}. \tag{10}$$

(It has been assumed that the sites of the system are identical, that is, the system is translation-invariant. Otherwise H in the right-hand side of (10) would depend on i.) The two-site Hamiltonian H is stochastic, that is, its nondiagonal elements are nonnegative, and the sum of the elements of each of its columns vanishes:

$$(\mathbf{s} \otimes \mathbf{s}) H = 0. \tag{11}$$

Here H is a 4×4 matrix (as the system under consideration has two possible states in each site and the interactions are nearest neighbor). The non-diagonal elements of H are nonnegative and equal to the interaction rates; that is, the element H_β^α with $\alpha \neq \beta$ is equal to the rate of change of the state β to the state α. α and β, each represent the state of two adjacent sites. For example if $\alpha = 01$ and $\beta = 10$, then H_β^α is the rate of particle diffusion to the right.

Using

$$\mathbf{s} \otimes \mathbf{s}(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{b} \cdot \mathbf{N}) H = a_\alpha\, b_\beta\, H^{\alpha\beta}{}_{\gamma\delta}\, \mathbf{s} \otimes \mathbf{s} N^\gamma \otimes N^\delta, \tag{12}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary covectors, one can write down the evolution equations of the one- and two-point functions of n_i's. It turns out that in the evolution equation of the one-point function, there are two-point functions, and in the evolution-equation of the two-point function, there are three point functions, unless the reaction rates satisfy the following constraints [27–29]

$${}^e\mathcal{A}^\alpha{}_{\gamma\delta} = {}^e_1\mathcal{A}^\alpha{}_\gamma\; s_\delta + {}^e_2\mathcal{A}^\alpha{}_\delta\; s_\gamma, \tag{13}$$

where

$$\begin{aligned} {}^1\mathcal{A}^\alpha{}_{\gamma\delta} &:= s_\beta\, H^{\alpha\beta}{}_{\gamma\delta} \\ {}^2\mathcal{A}^\alpha{}_{\gamma\delta} &:= s_\beta\, H^{\beta\alpha}{}_{\gamma\delta}. \end{aligned} \tag{14}$$

It can be seen that one can summarize the constraints (13) in the compact form

$$H\, \mathbf{u} \otimes \mathbf{u} = \lambda\, \mathbf{u} \otimes \mathbf{u}, \tag{15}$$

where

$$\mathbf{u} := \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{16}$$

and it is obvious that

$$\mathbf{s} \cdot \mathbf{u} = 0. \tag{17}$$

Now, consider a system satisfying the constraints (13) (or equivalently (15)), and take the vector $\mathbf{v}$ satisfying

$$\left(\sum_{d,e=1}^{2} {}^d_e\mathcal{A} \right) \mathbf{v} = 0,$$

$$\mathbf{s} \cdot \mathbf{v} = 1, \tag{18}$$

and the covector $\mathbf{a}$ such that

$$\mathbf{a} \cdot \mathbf{u} = 1, \qquad \mathbf{a} \cdot \mathbf{v} = 0, \tag{19}$$

that is, the basis $\{\mathbf{a},\mathbf{s}\}$ is dual to $\{\mathbf{u},\mathbf{v}\}$. This choice of $\mathbf{a}$ makes the evolution equation of $\langle \mathbf{a}\cdot\mathbf{N}\rangle$ homogeneous. In [28,29], it is shown that the matrix in the left-hand side of the first equation in (18), has a left eigenvector with the eigenvalue zero. (This left eigenvector is $\mathbf{s}$.) So it does have a right eigenvector with the eigenvalue zero as well. That is, there does exist a vector $\mathbf{v}$ satisfying (18). In fact, one can even find a real vector $\mathbf{v}$ satisfying (18). From now on, $\mathbf{a}$ in (6) is assumed to satisfy (19).

Assume further, that the initial condition is translational-invariant. This means that the one-point function is independent of the site, and the two-point function depends on only the difference of the sites' numbers. It turns out that the evolution equation for the one-point function is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = (\mu+\nu)f, \tag{20}$$

where

$$f := \langle n_i\rangle, \tag{21}$$

and

$$\begin{aligned}\mu =&\mathbf{s}\otimes\mathbf{a}\,H\,\mathbf{u}\otimes\mathbf{v}+\mathbf{a}\otimes\mathbf{s}\,H\,\mathbf{v}\otimes\mathbf{u},\\ \nu =&\mathbf{s}\otimes\mathbf{a}\,H\,\mathbf{v}\otimes\mathbf{u}+\mathbf{a}\otimes\mathbf{s}\,H\,\mathbf{u}\otimes\mathbf{v}.\end{aligned} \tag{22}$$

Also, taking

$$F_i := \langle n_k\, n_{k+i}\rangle, \tag{23}$$

one arrives at

$$\begin{aligned}\frac{\mathrm{d}F_i}{\mathrm{d}t} &= \mu(F_{i-1}+F_{i+1})+2\nu\,F_i, \qquad 1<i<L\\ \frac{\mathrm{d}F_1}{\mathrm{d}t} &= \mu\,F_2+(\nu+\lambda)F_1+\rho\,f+\sigma,\end{aligned} \tag{24}$$

where

$$\begin{aligned}\rho :=&\mathbf{a}\otimes\mathbf{a}\,H\,(\mathbf{u}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}),\\ \sigma :=&\mathbf{a}\otimes\mathbf{a}\,H\,\mathbf{v}\otimes\mathbf{v}.\end{aligned} \tag{25}$$

From the definition (23), it is also seen that

$$F_{L+1-i} = F_i. \tag{26}$$

It is seen that only five parameters enter the evolution equation of the up-to-two-point functions, and all of these can be expressed in terms of the matrix elements of

$$\bar{H} := H + \Pi\,H\,\Pi, \tag{27}$$

where Π is the permutation matrix. These parameters can be rewritten as

$$\begin{aligned}\mu :=&\mathbf{s}\otimes\mathbf{a}\,\bar{H}\,\mathbf{u}\otimes\mathbf{v}\\ \nu :=&\mathbf{s}\otimes\mathbf{a}\,\bar{H}\,\mathbf{v}\otimes\mathbf{u}\\ \lambda :=&\frac{1}{2}\mathbf{a}\otimes\mathbf{a}\,\bar{H}\,\mathbf{u}\otimes\mathbf{u}\\ \rho :=&\mathbf{a}\otimes\mathbf{a}\,\bar{H}\,\mathbf{u}\otimes\mathbf{v}\\ \sigma :=&\frac{1}{2}\mathbf{a}\otimes\mathbf{a}\,\bar{H}\,\mathbf{v}\otimes\mathbf{v}.\end{aligned} \tag{28}$$

3 Solution of the evolution equations

The solution to (20) (the evolution equation of the one-point function) is easily seen to be

$$f(t) = f(0)\exp[(\mu+\nu)t]. \tag{29}$$

Putting this in (24), the second equation becomes

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \mu\,F_2+(\nu+\lambda)F_1+\rho\,f(0)\exp[(\mu+\nu)t]+\sigma. \tag{30}$$

This, combined with the first equation of (24), and the constraint (26), are sufficient to obtain the two-point functions from their initial value. To do so, one takes a solution like

$$F_i(t) = \sum_E F_{i\,E}(0)\exp(E\,t), \tag{31}$$

and puts it in the equations. From the first equation of (24), one arrives at

$$E\,F_{i\,E}(0) = \mu[F_{i-1\,E}(0)+F_{i+1\,E}(0)]+2\nu\,F_{i\,E}(0), \qquad 1<i<L. \tag{32}$$

(30) becomes

$$EF_{1\,E}(0) = \mu F_{2\,E}(0)+(\nu+\lambda)F_{1\,E}(0)+\rho f(0)\,\delta_{\mu+\nu,E}+\sigma\,\delta_{0,E}. \tag{33}$$

To solve (32), one takes

$$F_{i\,E}(0) = c_E\,z^i + d_E\,z'^{\,i}. \tag{34}$$

Putting this in (32), one arrives at

$$E = \mu(z+z^{-1})+2\nu. \tag{35}$$

The equation for z' is similar, and in fact z' is the inverse of z. Then, using (26), one can write (34) as

$$F_{i\,E}(0) = c_E\left(z^i + z^{L+1-i}\right). \tag{36}$$

Putting this in (33), one can obtain the coefficient c_E.

For $E=\mu+\nu$, or $E=0$, the equation for c_E is a nonhomogeneous one, and c_E is obtained. For E different to the above values, the equation for c_E is a homogeneous one, and only for certain values of E there exist nonzero solutions for c_E. These values of E are among the eigenvalues of $\mathcal{H}$, of course. Using (36, 33), and (35), the condition for c_E being nonzero is seen to be

$$\mu\left[z^{-(L+1)/2}+z^{(L+1)/2}\right] = (\lambda-\nu)\left[z^{-(L-1)/2}+z^{(L-1)/2}\right]. \tag{37}$$

Some of the roots of this equation (for z) are phases (their

absolute value is one). In the thermodynamic limit (L to infinity) it is easy to find the nonphase solutions. It is seen that if a solution has absolute value less than one, then in the thermodynamic limit,

$$z = \frac{\mu}{\lambda - \nu}, \tag{38}$$

and it is obvious that such roots of (37) are real. So, in the thermodynamic limit, the energy values E entering the translationally-invariant two point function are zero, $(\mu + \nu)$, the values coming from (35) with $|z| = 1$, and possibly only one other value coming from (35) with z satisfying (38). The largest nonzero value of E, determines the relaxation time towards the equilibrium.

One point should be noted. In general, the limit of the largest relaxation time of a finite system, as its size tends to infinity, may differ from the relaxation time of the infinite system. It can be shown, however, that it is not the case for our system. In fact, if one solves the eigenvalue equation for the infinite system, one has to omit the periodicity condition (26), and use instead a condition that the two-point function does not blow up in the limit that the distance between the two points tends to infinity. This means that either z and z' (the inverse of z) are unimodular, or in (34) there remains only one term, the term corresponding to the one with modulus less than one. For the latter case, one again recovers (38). So, the spectrum of the infinite system, is in fact equal to the limit of the spectrum of the finite system with periodic boundary conditions, in the infinite-size limit. It is of course true that if in the initial condition, the coefficients of some eigenvectors vanish, then the relaxation time may differ from the largest relaxation time. But this happens independent of the size of the system. Another case when the relaxation behavior of the infinite system differs from the limit that of the finite system, is when the spectrum becomes continuous to zero, that is, in the infinite limit system, there is no eigenvalue gap between zero and the other part of the spectrum. In this case, the relaxation behavior of the infinite system may be a power law, rather than exponential decaying. But again this is not the case for the present system. To summarize, in the present system the largest relaxation time of the infinite system is equal to the limit of the largest relaxation time of the finite system.

4 The parameter space determining the energy-spectrum of the two-point functions

Consider a one-dimensional single-species nearest-neighbor-interacting system, for which the evolution equations of up-to-n-point functions are closed (we call such systems autonomous). The Hamiltonian H characterizing such a system (hence satisfying (15)), contains 10 parameters. As it was seen from the previous section, of the parameters entering H, only five parameters enter in the evolution equation of the two-point functions. All of these are expressible in terms of the symmetrized (with respect to permutation) Hamiltonian $\bar{H}$. It is easily seen that as H satisfies (15), $\bar{H}$ satisfies (15) as well. So the system characterized by $\bar{H}$, is autonomous as well. Such a system contains 6 independent rates. In fact, one can write $\bar{H}$ as

$$\bar{H} = \begin{pmatrix} -2r_1 - r_2 & r_3 & r_3 & r_5 \\ r_1 & -r_3 - r_7 - r_4 & r_7 & r_6 \\ r_1 & r_7 & -r_3 - r_7 - r_4 & r_6 \\ r_2 & r_4 & r_4 & -r_5 - 2r_6 \end{pmatrix}, \tag{39}$$

with

$$r_1 + r_2 + r_3 = r_4 + r_5 + r_6. \tag{40}$$

Of the five parameters entering the evolution equation of the two-point function, only three parameters determine the energy-spectrum. These are μ, ν, and λ:

$$\begin{aligned} \mu &= r_7 + r_4 - r_1 - r_2 = r_7 + r_3 - r_5 - r_6, \\ \nu &= -r_7 - r_1 - r_2 - r_3 = -r_7 - r_4 - r_5 - r_6, \\ \lambda &= -\frac{r_1 + r_3 + r_4 + r_6}{2}. \end{aligned} \tag{41}$$

From these relations, it is seen that

$$\begin{aligned} \nu &\le -|\mu| \le 0, \\ \nu &\le \lambda \le 0. \end{aligned} \tag{42}$$

As the rates are nonnegative, if $\nu = 0$, then $\bar{H} = 0$, which makes the two-point functions constant. Assuming $\nu \neq 0$, one can scale time and make $\nu = -1$. So, apart from a time-scale, there are only two parameters determining the energy-spectrum, μ and λ:

$$\begin{aligned} |\mu| &\le 1, \\ -1 &\le \lambda \le 0, \\ \nu &= -1. \end{aligned} \tag{43}$$

It can be shown that the whole region of the above is physical. That is, corresponding to any λ and μ satisfying the above inequalities, there are autonomous systems yielding the desired λ and μ. To prove this, first consider four specific systems:

- $r_1 = r_6 = 1, \quad r_2 = r_3 = r_4 = r_5 = r_7 = 0, \quad \Rightarrow \quad (\mu, \lambda) = (-1, -1).$
- $r_2 = r_5 = 1, \quad r_1 = r_3 = r_4 = r_6 = r_7 = 0, \quad \Rightarrow \quad (\mu, \lambda) = (-1, 0).$
- $r_3 = r_4 = 1, \quad r_1 = r_2 = r_5 = r_6 = r_7 = 0, \quad \Rightarrow \quad (\mu, \lambda) = (1, -1).$
- $r_7 = 1, \quad r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0, \quad \Rightarrow \quad (\mu, \lambda) = (1, 0).$

Any point (μ, λ) in the region described by (43), can be written as

$$(\mu, \lambda) = c_1(-1, -1) + c_2(-1, 0) + c_3(1, -1) + c_4(1, 0), \tag{44}$$

where c_a's are nonnegative. A system with the rates

$$r_i = \sum_{a=1}^{4} c_a \, r_{ai}, \tag{45}$$

where r_{ai} is the rate r_i of the a'th system introduced above, gives the desired (μ, λ).

5 Dynamic phase transitions in the two-point function

It was shown in the previous section, that for any nonzero Hamiltonian $\nu \neq 0$, so that one can normalize ν to -1. In section 3, it was shown that the energies entering the two-point function are 0, $E_1 := \mu - 1$, and $\mu(z+z^{-1})-2$, where in the thermodynamic limit $|z| = 1$, or at most one non-unimodular z exists, the value of which comes from (38). This is provided the absolute-value of the left-hand side of (38) is less than one. So, the energies (apart from 0) are E_1, any number in the interval $I_0 := [-2-2|\mu|, -2+2|\mu|]$, and possibly

$$E_2 := \lambda - 1 + \frac{\mu^2}{\lambda + 1}. \tag{46}$$

The largest relaxation time of the two-point function is $-E_{\max}^{-1}$, where $E_{\max}$ is the largest nonzero value of the energy spectrum. The relaxation time of the one-point function is $-E_1^{-1}$. The fact that the energy spectrum of the one-point function consists of a single value, is a result of the translational-invariance of the initial state of the system. Otherwise, there would be many energies for the one-point function, which could lead to a dynamical phase-transition in the one-point function [30–33]. The comparison of the relaxation times of the two-point- and the one-point-functions, is a comparison of $E_{\max}$ and E_1. If the former is larger, the largest relaxation time of the two-point function is larger than the relaxation time of the one-point function (the slow phases). If the two are equal, the relaxation-times are equal (the fast phases). So, the relation of E_1, E_2, and I_0, determines the relaxation behavior of the two-point function (its dynamical phase). It is seen that

$$\begin{aligned} &I_0 < E_1, \qquad \mu > -\frac{1}{3}, \\ &E_1 \in I_0, \qquad \mu < -\frac{1}{3}, \end{aligned} \tag{47}$$

where $|\mu| \leq 1$ has also been used. If $E_1 = E_{\max}$, then the relaxation time of the one-point function is equal to the largest relaxation time of the two-point function. If $E_1 < E_{\max}$, the the largest relaxation time of the two-point function is larger.

For E_2 to be among the energies, the absolute value of the left-hand side of (38) should be less than one. So,

$$\begin{aligned} &\nexists E = E_2, \qquad \lambda < |\mu| - 1, \\ &\exists E = E_2, \qquad \lambda > |\mu| - 1. \end{aligned} \tag{48}$$

Finally,

$$(\lambda+1)(E_2 - E_1) = \lambda^2 + \mu^2 - \lambda\,\mu + \lambda - \mu =: f(\mu, \lambda), \tag{49}$$

from which (using $\lambda + 1$ is nonnegative),

$$\begin{aligned} &E_2 < E_1, \qquad f(\mu, \lambda) < 0, \\ &E_2 > E_1, \qquad f(\mu, \lambda) > 0. \end{aligned} \tag{50}$$

$f = 0$ is an ellipse, the interior points of which correspond to $E_2 < E_1$, and the exterior points of which correspond to $E_2 > E_1$.

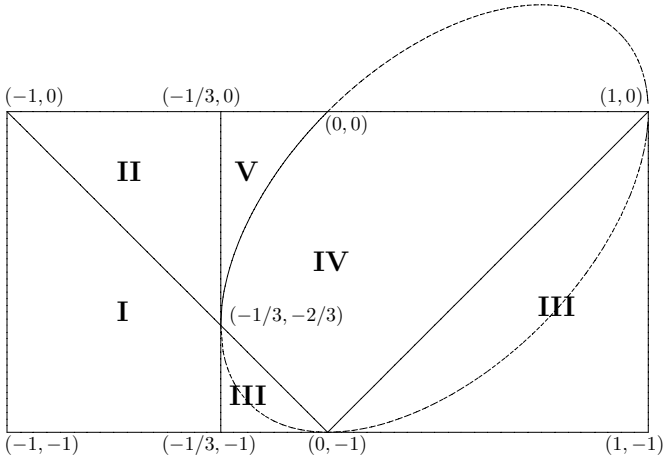

Fig. 1. The dynamical phase structure in the (μ, λ) plane.

These three inequalities divide the whole phase space ($|\mu| \leq 1$, $1 \leq \lambda \leq 0$) into five phases:

I) $\mu < -\frac{1}{3}, \quad \lambda < |\mu| - 1.$
In this phase, $E_1 \in I_0$, and E_2 is not an energy. This is the slower phase, and the largest energy is $E_{\max} = -2 - 2\mu$.

II) $\mu < -\frac{1}{3}, \quad \lambda > |\mu| - 1.$
In this phase, $E_1 \in I_0$, and E_2 is an energy, in fact the largest one. This is the slowest phase, and the largest energy is $E_{\max} = -1 + \lambda + \frac{\mu^2}{\lambda+1}$.

III) $\mu > -\frac{1}{3}, \quad \lambda < |\mu| - 1.$
In this phase, $E_1 > I_0$, and E_2 is not an energy. This is the fastest phase, and the largest energy is $E_{\max} = -1 + \mu$.

IV) $\mu > -\frac{1}{3}, \quad |\mu| - 1 < \lambda < \frac{\mu - 1 + \sqrt{(1+3\mu)(1-\mu)}}{2}$.
In this phase, $E_1 > I_0$, E_2 is an energy, and $E_2 < E_1$. This is the fast phase, and the largest energy is $E_{\max} = -1 + \mu$.

V) $\mu > -\frac{1}{3}, \quad \lambda > \frac{\mu - 1 + \sqrt{(1+3\mu)(1-\mu)}}{2}$.
In this phase, $E_1 > I_0$, E_2 is an energy, and $E_2 > E_1$. This is the slow phase, and the largest energy is $E_{\max} = -1 + \lambda + \frac{\mu^2}{\lambda+1}$.

This phase structure is summarized in Figure 1.

As previously mentioned, these phases arise from the fact that the energy-spectrum of the two-point function consists of a continuous part, an energy equal to the energy appearing in the one-point function, and possibly another energy. This shows that the relaxation of the two-point function is at least as slow as that of the one-point function, and may be slower, depending on the relative position of the discrete and continuous parts of the spectrum. The fast phases (phases **III** and **IV**), are those in them the relaxation time of the one- and two-point functions are equal, while in the slow phases (phases **I**, **II**, and **V**), the relaxation of the two-point function is slower than that of the one-point function.

6 A one-parameter family as an example

Consider a system with the Hamiltonian

$$H = \frac{1}{4}\begin{pmatrix} -3+3\omega & \omega & \omega & 1-\omega \\ 1-\omega & -3\omega & \omega & 1-\omega \\ 1-\omega & \omega & -3\omega & 1-\omega \\ 1-\omega & \omega & \omega & -3+3\omega \end{pmatrix}. \qquad (51)$$

This Hamiltonian describes a system with the following reactions.

$$\begin{aligned} \emptyset A \rightarrow & \text{ any other state}, & \text{with the rate } \omega/4, \\ A\emptyset \rightarrow & \text{ any other state}, & \text{with the rate } \omega/4, \\ \emptyset\emptyset \rightarrow & \text{ any other state}, & \text{with the rate } (1-\omega)/4, \\ AA \rightarrow & \text{ any other state}, & \text{with the rate } (1-\omega)/4. \end{aligned} \qquad (52)$$

It is seen that for this system,

$$\bar{H} = 2H, \qquad (53)$$

and

$$\begin{aligned} \mu &= -1+2\omega, \\ \lambda &= -\frac{1}{2}. \end{aligned} \qquad (54)$$

For this system, ρ defined through (28) is equal to zero, hence there is no term proportional to $e^{E_1 t}$ in the right-hand side of (30). However, it is seen that adding a term

$$H_1 = r\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \qquad (55)$$

to the Hamiltonian H in (51), changes the values of ρ, ν, μ, and λ to

$$\begin{aligned} \rho &= r, \\ \nu &= -1-2r, \\ \mu &= -1-2r+2\omega, \\ \lambda &= -\frac{1}{2}-r. \end{aligned} \qquad (56)$$

For small values of r, one can use $\nu = 1$, and μ and λ as in (54). Then, with different values of ω, this system can exist in all the above five phases:

$$\begin{aligned} &\text{phase } \mathbf{I}, && 0 \le \omega < \frac{1}{4}, \\ &\text{phase } \mathbf{II}, && \frac{1}{4} < \omega < \frac{1}{3}, \\ &\text{phase } \mathbf{V}, && \frac{1}{3} < \omega < \frac{5-\sqrt{5}}{8}, \\ &\text{phase } \mathbf{IV}, && \frac{5-\sqrt{5}}{8} < \omega < \frac{3}{4}, \\ &\text{phase } \mathbf{III}, && \frac{3}{4} < \omega \le 1. \end{aligned} \qquad (57)$$

It is seen that increasing ω, the system undergoes phase transitions from the phase **I** to **II**, then **V**, **IV**, and finally **III**.

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